Radial power-law position-dependent mass; Cylindrical coordinates, separability, and spectral signatures

Omar Mustafa

Department of Physics, Eastern Mediterranean University,

G Magusa, North Cyprus, Mersin 10,Turkey

E-mail: omar.mustafa@emu.edu.tr,

Tel: +90 392 630 1314,

Fax: +90 392 3651604

January 19, 2013

Abstract

We discuss the separability of the position-dependent mass Hamiltonian in cylindrical coordinates in the framework of a radial power-law position-dependent mass. We consider two particular radial mass settings; a harmonic oscillator type, and a Coulombic type. We subject the radial harmonic oscillator type mass to a radial harmonic oscillator potential and the radial Coulombic mass to a radial Coulombic potential. Azimuthal symmetry is assumed and spectral signatures of various z-dependent interaction potentials are reported.

PACS codes: 03.65.Ge, 03.65.Ca

Keywords: Power-law, Position-dependent-mass, cylindrical coordinates, separability, exact solvability, spectral signatures

1 Introduction

The von Roos Hamiltonian [1] is known to describe quantum particles with position-dependent-mass (PDM), $M(\vec{r}) = m_\circ m(\vec{r})$. Over the last few decades, the position-dependent-mass Hamiltonians have inspired research attention [2-32] because of their applicability in the study of many-body problem, semiconductors, quantum dots, quantum liquids, etc. The kinetic energy operator in the von Roos Hamiltonian (with $m_\circ = \hbar = 1$ units)

$$H = -\frac{1}{4}\left[m(\vec{r})^{\gamma}\,\vec{\nabla} m(\vec{r})^{\beta} \cdot \vec{\nabla} m(\vec{r})^{\alpha} + m(\vec{r})^{\alpha}\;\vec{\nabla} m(\vec{r})^{\beta} \cdot \vec{\nabla} m(\vec{r})^{\gamma}\right] + V(\vec{r}), \quad (1)$$

admits an ordering ambiguity manifested by the non-uniqueness representation of the kinetic energy operator. Which would, in effect, introduce a profile change in the effective potential as the values of the parameters α, β, and γ change [cf., e.g., 25-29]. Here, α, β, and γ are called the von Roos ordering ambiguity parameters satisfying the von Roos constraint $\alpha + \beta + \gamma = -1$. Nevertheless, an interesting and comprehensive background on the issue of consistency and usefulness of the position-dependent mass Schrödinger equation was given by Lèvy-Leblond [32]. Therein, his work is devoted to sustaining and strengthening the conclusions that not only the use of position-dependent mass gives correct approximation, but its also a conceptually consistent approach.

It is however constructive to mention that the continuity conditions at the abrupt heterojunction between two crystals implied that $\alpha = \gamma$, otherwise for $\alpha \neq \gamma$ the wavefunctions vanish at the boundaries and the heterojunction plays the role of impenetrable barrier (cf., e.g., Mustafa and Mazharimousavi [10]

and Koc et al. in [28]). Eliminating in the process, the Gora's and Williams' ($\beta = \gamma = 0$, $\alpha = -1$), and Li's and Kuhn's ($\beta = \gamma = -1/2$, $\alpha = 0$) known parametric sets. Moreover, Dutra's and Almeida's [9] reliability test classifies the parametric set of Ben Daniel and Duke ($\alpha = \gamma = 0$, $\beta = -1$) as a set to-be-discarded for it yields imaginary eigenvalues. This would leave us with Zhu's and Kroemer's ($\alpha = \gamma = -1/2$, $\beta = 0$) and Mustafa's and Mazharimousavi's ($\alpha = \gamma = -1/4$, $\beta = -1/2$) ordering ambiguity parameters that are classified as "good" parametric sets, so to speak. Nevertheless, we have observed (cf., e. g., [29]) that the physical and/or mathematical admissibility of a given ambiguity parametric set depends also on the form of the position-dependent-mass and/or the form of the interaction potential. In the forthcoming methodical proposal, we shall work with the ambiguity parameters as they are without any classification as to which set is "good" or "to-be-discarded".

Very recently, Mustafa [31] has considered the von Roos Hamiltonian (1) using cylindrical coordinates. Therein, we sought some manifestly feasible separability through the suggestion that the position-dependent-mass is only radial-dependent (i.e., $m(\vec{r}) = m_{\circ} M(\rho, \varphi, z) = M(\rho, \varphi, z) = M(\rho) = 1/\rho^2$), where azimuthal symmetrization is granted through a proper assumption of the interaction potential. The spectral signatures of different z-dependent interaction potential settings on the radial Coulombic and radial harmonic oscillator interaction potentials' spectra are reported for impenetrable walls at $z = 0$ and $z = L$, for a Morse, for a non-Hermitian $\mathcal{PT}$-symmetrized Scarf II, and for a non-Hermitian $\mathcal{PT}$-symmetrized Samsonov interaction models.

In this work, we offer a parallel azimuthal symmetrization along with a more general (though still only radial-dependent) power-law-type position-dependent-mass (i.e., $M(\rho, \varphi, z) = M(\rho) = b\rho^{2\upsilon+1}/2$). Obviously, a $\upsilon = -3/2$ and $b = 2$ yield $M(\rho) \sim 1/\rho^2$ which is, under the current forthcoming settings, a special

case of $M(\rho) = b\rho^{2\upsilon+1}/2$ that shall not be repeated here. Instead, we shall use $\upsilon = -1$ and $\upsilon = 1/2$ that yield quantum particles endowed with position-dependent masses of a Coulombic-type, $M(\rho) = b\rho^{-1}/2$, and a harmonic oscillator type, $M(\rho) = b\rho^{2}/2$, respectively. To the best of our knowledge, such position-dependent mass settings have not been considered elsewhere.

To make this work self-contained, we recollect (in section 2) the most relevant and vital relations (namely, equations (2)-(5) below) that have been readily reported in [31] for cylindrical coordinates separability and exact solvability of the PDM-Hamiltonian (1). In the same section, we discuss the separability in the framework of a manifestly radial power-law position-dependent mass and contemplate on the feasible separabilities. In section 3, we consider two particular radial mass settings; a harmonic oscillator type, $M(\rho,\varphi,z) = M(\rho) = g(\rho) = b\rho^{2}/2$, and a Coulombic type, $M(\rho,\varphi,z) = M(\rho) = g(\rho) = b\rho^{-1}/2$. We subject the radial harmonic oscillator type mass to move under the influence of a radial harmonic oscillator potential field $\tilde{V}(\rho) = a^{2}\rho^{2}/4$ and the radial Coulombic mass to a radial Coulombic potential $\tilde{V}(\rho) = -2\tilde{A}/\rho$. The spectral signatures of (i) two impenetrable walls at $z = 0$ and $z = L$ provided by the potential well $\tilde{V}(z) = 0$ for $0 < z < L$ and $\tilde{V}(z) = \infty$ elsewhere, (ii) a Morse type [31] interaction $\tilde{V}(z) = D\left(e^{-2\epsilon z} - 2e^{-\epsilon z}\right); D > 0$, and (iii) a trigonometric Rosen-Morse [32] potential $\tilde{V}(z) = U_{\circ}\cot^{2}(\pi z/d); z \in [0,d]$, are reported in the same section. Our concluding remarks are given in section 4.

2 Cylindrical coordinates and radial power-law PDM framework

Following our recent work [31] on cylindrical coordinates separability and exact solvability of the PDM-Hamiltonian (1), we again consider the position-

dependent-mass and the interaction potential to take the forms $m(\vec{r}) \equiv M(\rho,\varphi,z) = g(\rho)\,f(\varphi)\,k(z)$ and $V(\vec{r}) \equiv V(\rho,\varphi,z)$, respectively. We have shown (see Mustafa [31] for more details on this issue) that the corresponding PDM-Schrödinger equation $[H-E]\,\Psi(\rho,\varphi,z)=0$ with

$$\Psi(\rho,\varphi,z) = R(\rho)\,\Phi(\varphi)\,Z(z)\,;\;\; \rho\in(0,\infty)\,,\;\; \varphi\in(0,2\pi)\,, z\in(-\infty,\infty)\,, \tag{2}$$

would imply

$$\begin{aligned} 0 \;\; = \;\; & 2g(\rho)\,f(\varphi)\,k(z)\,[E-V(\rho,\varphi,z)] \\ & +\left[\frac{R''(\rho)}{R(\rho)} - \left(\frac{g'(\rho)}{g(\rho)} - \frac{1}{\rho}\right)\frac{R'(\rho)}{R(\rho)}\right. \\ & \left. +\frac{\zeta}{2}\left(\frac{g'(\rho)}{g(\rho)}\right)^2 - \frac{(\beta+1)}{2}\left(\frac{g'(\rho)}{\rho g(\rho)} + \frac{g''(\rho)}{g(\rho)}\right)\right] \\ & +\left[\frac{Z''(z)}{Z(z)} - \frac{k'(z)}{k(z)}\frac{Z'(z)}{Z(z)} + \frac{\zeta}{2}\left(\frac{k'(z)}{k(z)}\right)^2 - \frac{(\beta+1)}{2}\frac{k''(z)}{k(z)}\right] \\ & +\frac{1}{\rho^2}\left[\frac{\Phi''(\varphi)}{\Phi(\varphi)} - \frac{f'(\varphi)}{f(\varphi)}\frac{\Phi'(\varphi)}{\Phi(\varphi)} + \frac{\zeta}{2}\left(\frac{f'(\varphi)}{f(\varphi)}\right)^2 - \frac{(\beta+1)}{2}\frac{f''(\varphi)}{f(\varphi)}\right] \end{aligned} \tag{3}$$

where

$$\zeta = \alpha(\alpha-1) + \gamma(\gamma-1) - \beta(\beta+1)\,. \tag{4}$$

To facilitate and ease separability, we have suggested that the interaction potential satisfies an obviously "manifested-by-equation (3)" general identity of the form

$$2MV(\rho,\varphi,z) = 2g(\rho)\,f(\varphi)\,k(z)\,V(\rho,\varphi,z) = \tilde{V}(\rho) + \tilde{V}(z) + \frac{1}{\rho^2}\tilde{V}(\varphi)\,. \tag{5}$$

Hereby, we may remind the reader that in [31} we have used $g(\rho)=1/\rho^2$ along with $f(\varphi)=1=k(z)$ as one of the options that secured separability of the problem at hand.

In the search for a more general recipe, however, we choose to eliminate the first-order derivatives $Z'(z)$, $\Phi'(\varphi)$, and $R'(\rho)$. At this point, the elimination of the first-order derivatives of $Z(z)$ and $\Phi(\varphi)$ is achieved through the substitutions

$$Z(z)=\sqrt{k(z)}\tilde{Z}(z) \text{ and } \Phi(\varphi)=\sqrt{f(\varphi)}\tilde{\Phi}(\varphi), \tag{6}$$

to imply that

$$\frac{Z''(z)}{Z(z)}-\frac{k'(z)}{k(z)}\frac{Z'(z)}{Z(z)}=-\frac{3}{4}\left(\frac{k'(z)}{k(z)}\right)^2+\frac{1}{2}\frac{k''(z)}{k(z)}+\frac{\tilde{Z}''(z)}{\tilde{Z}(z)} \tag{7}$$

and

$$\frac{\Phi''(\varphi)}{\Phi(\varphi)}-\frac{f'(\varphi)}{f(\varphi)}\frac{\Phi'(\varphi)}{\Phi(\varphi)}=-\frac{3}{4}\left(\frac{f'(\varphi)}{f(\varphi)}\right)^2+\frac{1}{2}\frac{f''(\varphi)}{f(\varphi)}+\frac{\tilde{\Phi}''(\varphi)}{\tilde{\Phi}(\varphi)} \tag{8}$$

Whereas, the elimination of the first-order derivative of $R(\rho)$ may be sought through the substitutions

$$R(\rho)=\rho^{\upsilon}U(\rho) \text{ and } g(\rho)=\frac{b}{2}\rho^{2\upsilon+1};\upsilon,b\in\mathbb{R}, \tag{9}$$

(with the restriction that b is a non-zero constant to avoid triviality) to imply that

$$\frac{R''(\rho)}{R(\rho)}-\left(\frac{g'(\rho)}{g(\rho)}-\frac{1}{\rho}\right)\frac{R'(\rho)}{R(\rho)}=\frac{U''(\rho)}{U(\rho)}-\frac{\upsilon(\upsilon+1)}{\rho^2}. \tag{10}$$

It should be noted here that the choice of $g(\rho)=b\rho^{2\upsilon+1}/2$ in (9) is manifestly mandated by the elimination of the first-order derivative of $U(\rho)$.

Under such settings, equation (3) would read

$$
\begin{aligned}
0 &= b\rho^{2\upsilon+1} f(\varphi)\, k(z)\, E \\
&+ \left[\frac{U''(\rho)}{U(\rho)} + \frac{(2\upsilon+1)^2 [\zeta-\beta-1] - 2\upsilon(\upsilon+1)}{2\rho^2} - \tilde{V}(\rho)\right] \\
&+ \left[\frac{\tilde{Z}''(z)}{\tilde{Z}(z)} + \frac{(2\zeta-3)}{4}\left(\frac{k'(z)}{k(z)}\right)^2 - \frac{\beta}{2}\frac{k''(z)}{k(z)} - \tilde{V}(z)\right] \\
&+ \frac{1}{\rho^2}\left[\frac{\tilde{\Phi}''(\varphi)}{\tilde{\Phi}(\varphi)} + \frac{(2\zeta-3)}{4}\left(\frac{f'(\varphi)}{f(\varphi)}\right)^2 - \frac{\beta}{2}\frac{f''(\varphi)}{f(\varphi)} - \tilde{V}(\varphi)\right]. \qquad (11)
\end{aligned}
$$

Hereby, it should mentioned that the case where $\upsilon = -3/2$, $b = 2$ (i.e., $g(\rho) = 1/\rho^2$) and $k(z) = 1 = f(\varphi)$ is the case we have considered in [31]. It is now just a special case of the current, though more general, recipe $g(\rho) = b\rho^{2\upsilon+1}/2; \upsilon, b \in \mathbb{R}$. The results and examples reported therein are recoverable and hold true for the current work, therefore.

Yet, an obvious manifestation of the energy term $b\rho^{2\upsilon+1} f(\varphi)\, k(z)\, E$ towards separability of (11) is that, in addition to the three feasible separable cases $f(\varphi) = 1 = k(z)$, $k(z) = 1 = g(\rho)$, and $f(\varphi) = 1 = g(\rho)$ (reported in [31]), one finds two more feasibly separable cases. They are, for $\upsilon = -3/2$, $k(z) = 1$, $f(\varphi) \neq 1$ (which, in turn, would break azimuthal symmetry) and for $\upsilon = -3/2$, $k(z) \neq 1$, $f(\varphi) = 1$. Therefore, the separability of (11) may be facilitated by the forms of the position-dependent mass and of the interaction potential $V(\rho, \varphi, z)$.

To secure azimuthal symmetrization of the problem at hand we substitute $\tilde{V}(\varphi) = 0$ and $f(\varphi) = 1$. Moreover, we choose $k(z) = 1$ to imply that

$$
\frac{\tilde{\Phi}''(\varphi)}{\tilde{\Phi}(\varphi)} = k_\varphi^2 \ ; \ k_\varphi^2 = -m^2 \ ; \ |m| = 0, 1, 2, \cdots, \qquad (12)
$$

$$
\left[-\partial_z^2 + \tilde{V}(z)\right] \tilde{Z}(z) = k_z^2\, \tilde{Z}(z), \qquad (13)
$$

and

$$\left[-\partial_\rho^2+\frac{\tilde{\ell}_\upsilon^2-1/4}{\rho^2}+\tilde{V}\left(\rho\right)-b\rho^{2\upsilon+1}E\right]U\left(\rho\right)=-k_z^2U\left(\rho\right). \tag{14}$$

Where m is the magnetic quantum number and

$$\left|\tilde{\ell}_\upsilon\right|=\sqrt{\upsilon\left(\upsilon+1\right)+m^2+\frac{1}{4}-\frac{\left(2\upsilon+1\right)^2\left[\zeta-\beta-1\right]}{2}}. \tag{15}$$

is an irrational magnetic quantum number. Hereby, it obvious that the substitutions of $\upsilon=-3/2$ and $b=2$ in (14) would inspire a re-scale of the form

$$\tilde{\ell}_{-3/2}^2=\ell^2+2E=\left(m^2+3\right)-2\left(\zeta-\beta\right)+2E, \tag{16}$$

so that our results in [31] for the radial Coulombic $\tilde{V}\left(\rho\right)=-2/\rho$ and the radial harmonic oscillator $\tilde{V}\left(\rho\right)=a^2\rho^2/4$ (equations (23) and (24) in [31], respectively) are safely reproduced. Therefore, the corresponding spectral signatures of $\tilde{V}\left(z\right)$ interaction potentials for $\upsilon=-3/2$ and $b=2$ (i.e, $g\left(\rho\right)=\rho^{-2}$) are reported therein [31] and shall not be repeated here.

3 Two particular radial settings; $g\left(\rho\right)=b\rho^2/2$ and $g\left(\rho\right)=b\rho^{-1}/2$

In this section. we consider the position-dependent mass $M\left(\rho,\varphi,z\right)=g\left(\rho\right)$ to indulge a radial harmonic oscillator $g\left(\rho\right)=b\rho^2/2$ (i.e., $\upsilon=1/2$) and the radial Coulombic $g\left(\rho\right)=b\rho^{-1}/2$ (i.e., $\upsilon=-1$) forms. For the sake of keeping this work simple and instructive, we shall consider the radial harmonic oscillator $g\left(\rho\right)=b\rho^2/2$ accompanied by a radial harmonic oscillator type interaction $\tilde{V}\left(\rho\right)=a^2\rho^2/4$ and the radial Coulombic $g\left(\rho\right)=b\rho^{-1}/2$ accompanied by a radial Coulombic $\tilde{V}\left(\rho\right)=-2\,\tilde{A}/\rho$. We shall moreover report the spectral signa-

tures of different $\tilde{V}(z)$ potentials on the overall spectrum.

3.1 The radial harmonic oscillator $g(\rho) = b\rho^2/2$

The choice of $\upsilon = 1/2$ along with $\tilde{V}(\rho) = a^2\rho^2/4$ would imply that equation (14) reads

$$\left[-\partial_\rho^2 + \frac{\tilde{\ell}_{1/2}^2 - 1/4}{\rho^2} + \frac{\left(a^2 - 4bE\right)}{4}\rho^2\right] U(\rho) = -k_z^2 U(\rho), \tag{17}$$

and (15), in turn, yields

$$\left|\tilde{\ell}_{1/2}\right| = \sqrt{(m^2+3) - 2(\zeta - \beta)}. \tag{18}$$

Obviously, Eq.(17) has exact eigenvalues in the form

$$k_z^2 = -\sqrt{(a^2 - 4bE)}\left[2n_\rho + \left|\tilde{\ell}_{1/2}\right| + 1\right]^2, \tag{19}$$

and implies that

$$E = \frac{a^2}{4b} - \frac{1}{4b}\left[\frac{k_z^2}{2n_\rho + \sqrt{(m^2+3) - 2(\zeta - \beta)} + 1}\right]^2. \tag{20}$$

We observe that an auxiliary constraint

$$(\zeta - \beta) = \alpha(\alpha - 1) + \gamma(\gamma - 1) - \beta(\beta + 2) \leq \left(m^2 + 3\right)/2 \tag{21}$$

on the ambiguity parameters is manifested here by the requirement that $E \in \mathbb{R}$.

3.1.1 Spectral signatures of some $\tilde{V}(z)$ potentials on the radial harmonic oscillator spectrum

Recollect [31] that if our PDM-particle is trapped to move between two impenetrable walls at $z=0$ and $z=L$ under the influence of a

$$\tilde{V}(z)=\begin{cases} 0 & ;\, 0<z<L \\ \infty & ;\, \text{elsewhere} \end{cases}, \tag{22}$$

one would find that $K_z = n_z\pi/L$, $n_z = 1,2,3,\cdots$ (see [31] for more details on this issue). This would, in effect, give the spectral signature of $\tilde{V}(z)$ of (22) on the overall spectrum

$$E=\frac{a^2}{4b}-\frac{1}{4b}\left[\frac{(n_z\pi/L)^2}{2n_\rho+\sqrt{(m^2+3)-2(\zeta-\beta)+1}}\right]^2 \tag{23}$$

for a PDM particle of $M(\rho,\varphi,z)=M(\rho)=b\rho^2/2$ moving in a potential of the form

$$V(\rho,\varphi,z)=\frac{a^2}{4b}+\begin{cases} 0 & ;\, 0<z<L \\ \infty & ;\, \text{elsewhere} \end{cases}. \tag{24}$$

Next, let us subject this PDM particle to move in a Morse type [31] interaction $\tilde{V}(z)=D\ \left(e^{-2\epsilon z}-2e^{-\epsilon z}\right);\, D>0$. In this case

$$k_z^2=\left(\frac{\sqrt{D}}{\epsilon}-\tilde{n}_z-\frac{1}{2}\right),\ \tilde{n}_z=0,1,2,3,\cdots \tag{25}$$

Therefore, a PDM quantum particle endowed with $M(\rho,\varphi,z)=M(\rho)=b\rho^2/2$ and subjected to an interaction potential of the form

$$V(\rho,\varphi,z)=\frac{a^2}{4b}+\frac{D}{b\rho^2}\ \left(e^{-2\epsilon z}-2e^{-\epsilon z}\right);D>0 \tag{26}$$

would admit exact energy eigenvalues given by

$$E=\frac{a^{2}}{4b}-\frac{1}{4b}\left[\frac{\left(\sqrt{D}/\epsilon-\tilde{n}_{z}-\frac{1}{2}\right)}{2n_{\rho}+\sqrt{(m^{2}+3)-2\left(\zeta-\beta\right)}+1}\right]^{2}. \tag{27}$$

Now, let $M(\rho,\varphi,z)=M(\rho)=b\rho^{2}/2$ move under the influence of a trigonometric Rosen-Morse potential $\tilde{V}(z)=U_{\circ}\cot^{2}(\pi z/d)\,;z\in[0,d]$. Where $U_{\circ}$ and d are two positive parameters. In this case,

$$V(\rho,\varphi,z)=\frac{a^{2}}{4b}+\frac{U_{\circ}}{b\rho^{2}}\ \cot^{2}(\pi z/d)\ ;\ z\in[0,d]\,, \tag{28}$$

$$k_{z}^{2}=\frac{1}{d^{2}}\left[Cd+\tilde{n}_{z}\pi\right]^{2}-U_{\circ}\ ;\ C=\frac{\pi}{2d}\left(1+\sqrt{1+\frac{4U_{\circ}d^{2}}{\pi^{2}}}\right), \tag{29}$$

(see Ma et al [32] for more details, notice that one should consider $2\mu=\hbar=1$ of Ma as proper parametric mapping into our settings) and

$$E=\frac{a^{2}}{4b}-\frac{1}{4b}\left[\frac{\left[Cd+\tilde{n}_{z}\pi\right]^{2}/d^{2}-U_{\circ}}{2n_{\rho}+\sqrt{(m^{2}+3)-2\left(\zeta-\beta\right)}+1}\right]^{2}. \tag{30}$$

3.2 The Radial Coulombic $g(\rho)=b\rho^{-1}/2$

Now consider the PDM-particle to have a radial Coulombic-type mass of the form $M(\rho,\varphi,z)=M(\rho)=b\rho^{-1}/2$, (i.e., $\upsilon=-1$) and subjected to move in a radial Coulombic potential $\tilde{V}(\rho)=-2\tilde{A}/\rho$. In this case,

$$V(\rho,\varphi,z)=-\frac{\tilde{A}}{b}+\frac{\rho}{b}\tilde{V}(z)\,, \tag{31}$$

and equation (14) yields

$$\left[-\partial_{\rho}^{2}+\frac{\tilde{\ell}_{-1}^{2}-1/4}{\rho^{2}}-\frac{2\left(\tilde{b}E+\tilde{A}\right)^{2}}{\rho}\right]U(\rho)=-k_{z}^{2}U(\rho)\,;\,\tilde{b}=b/2, \tag{32}$$

with

$$\left|\tilde{\ell}_{-1}\right| = \sqrt{(m^2+3/4) - (\zeta-\beta)/2}, \tag{33}$$

and

$$k_z = \pm \frac{\tilde{b}E + \tilde{A}}{\left(n_\rho + \left|\tilde{\ell}_{-1}\right| + 1\right)}. \tag{34}$$

Which, in turn, results

$$E = \pm \frac{k_z}{\tilde{b}} \left(n_\rho + \sqrt{(m^2+3/4) - (\zeta-\beta)/2} + 1\right) - \frac{\tilde{A}}{\tilde{b}}, \tag{35}$$

with the auxiliary constraint

$$(\zeta-\beta) = \alpha(\alpha-1) + \gamma(\gamma-1) - \beta(\beta+2) \leq \left(2m^2+3/2\right), \tag{36}$$

on the ambiguity parameters that secures the reality of E. Nevertheless, two branches of energies are obviously obtained. Moreover, the spectral signature of k_z on the overall spectrum is obtained through the solution of equation (13).

3.2.1 Spectral signatures of some $\tilde{V}(z)$ potentials on the radial Coulombic spectrum

If we subject our radial Coulombic PDM-particle $M(\rho,\varphi,z) = M(\rho) = b\rho^{-1}/2$ to move in $V(\rho,\varphi,z) = -\tilde{A}/b + \rho\tilde{V}(z)/b$,where $\tilde{V}(z)$ is given by (22), it will admit exact energies of the form

$$E = \pm \frac{n_z\pi}{\tilde{b}L} \left(n_\rho + \sqrt{(m^2+3/4) - (\zeta-\beta)/2} + 1\right) - \frac{\tilde{A}}{\tilde{b}};\ n_z = 1,2,3,\cdots. \tag{37}$$

Moreover, if this PDM-particle is subjected to move in a Morse type [31]

interaction $\tilde{V}(z) = D\ \left(e^{-2\epsilon z} - 2e^{-\epsilon z}\right);\ D > 0$. In this case,

$$V(\rho, \varphi, z) = -\frac{\tilde{A}}{b} + \frac{\rho}{b} D\ \left(e^{-2\epsilon z} - 2e^{-\epsilon z}\right); D > 0, \tag{38}$$

and the exact energies are of the form

$$E = \pm\frac{1}{\tilde{b}}\sqrt{\left(\frac{\sqrt{D}}{\epsilon} - \tilde{n}_z - \frac{1}{2}\right)\left(n_\rho + \sqrt{(m^2 + 3/4) - (\zeta - \beta)/2} + 1\right)} - \frac{\tilde{A}}{\tilde{b}}, \tag{39}$$

Where $\tilde{n}_z = 0, 1, 2, 3, \cdots$. Obviously, the condition $\left(\sqrt{D}/\epsilon - \tilde{n}_z - \frac{1}{2}\right) > 0$ is manifested here and ought to be enforced, otherwise complex pairs of energy eigenvalues are obtained in the process.

Next, let $M(\rho, \varphi, z) = M(\rho) = b\rho^{-1}/2$ move under the influence of a trigonometric Rosen-Morse potential $\tilde{V}(z) = U_\circ \cot^2(\pi z/d); z \in [0, d]$. Then,

$$V(\rho, \varphi, z) = -\frac{\tilde{A}}{b} + \frac{\rho}{b} U_\circ \cot^2(\pi z/d); z \in [0, d], \tag{40}$$

and

$$E = \pm\frac{\sqrt{\left[Cd + \tilde{n}_z\pi\right]^2 - U_\circ d^2}}{\tilde{b}d}\left(n_\rho + \sqrt{(m^2 + 3/4) - (\zeta - \beta)/2} + 1\right) - \frac{\tilde{A}}{\tilde{b}}. \tag{41}$$

4 Concluding remarks

We have recollected the most relevant and vital relations (equations (2)-(5) above) that have been readily reported by Mustafa [31] for cylindrical coordinates separability of the PDM-Hamiltonian in (1), where the PDM-setting was considered in the form $M(\rho, \varphi, z) = g(\rho) f(\varphi) k(z) = g(\rho) = 1/\rho^2$ under azimuthally symmetric settings.

In this work, however, we offered a more general power-law radial position-

dependent mass recipe $M(\rho,\varphi,z)=g(\rho)=b\rho^{2\upsilon+1}/2;\upsilon,b\in\mathbb{R}$, within which $M(\rho,\varphi,z)=g(\rho)=1/\rho^2$ of [31] represents a special case (the results and examples reported therein hold true and yet document additional examples on the applicability of the current methodical proposal, therefore). Moreover, the structure of the position-dependent energy term $b\rho^{2\upsilon+1}f(\varphi)\,k(z)\,E$ in (11) suggests that there are five feasible cases towards separability; (i) $f(\varphi)=1=k(z)$, (ii) $k(z)=1=g(\rho)$, (iii) $f(\varphi)=1=g(\rho)$, (iv) $\upsilon=-3/2$, $k(z)\neq 1$, $f(\varphi)=1$, and (v) $\upsilon=-3/2$, $k(z)=1$, $f(\varphi)\neq 1$ (which would break azimuthal symmetry, of course). Therefore, the separability of (3) may be facilitated by the forms of the position-dependent mass and the interaction potential $V(\rho,\varphi,z)$. These are not the only cases to secure separability of (3), so to speak.

We have considered two particular mass settings; a radial harmonic oscillator type, $M(\rho,\varphi,z)=M(\rho)=g(\rho)=b\rho^2/2$, and a radial Coulombic type, $M(\rho,\varphi,z)=M(\rho)=g(\rho)=b\rho^{-1}/2$. We have observed that for the Coulombic case two branches of energies are obtained, each of which is a "mirror-reflection" of the other about the zero-energy axis. Moreover, when we subjected the radial harmonic oscillator mass to a radial harmonic oscillator potential $\tilde{V}(\rho)=a^2\rho^2/4$ and the radial Coulombic mass to a radial Coulombic potential $\tilde{V}(\rho)=-2\,\tilde{A}/\rho$, only constant shifts in the energies where observed (i.e., a shift $\left(a^2/4b\right)$ for the radial harmonic oscillator mass and $\left(-\tilde{A}/\tilde{b}\right)$ for the radial Coulombic mass, documented in (20) and (35), respectively). That is, the radial interaction potentials $\tilde{V}(\rho)$ considered for the two over simplified examples here provided no quantization recipe at all (i.e., they have only introduced constant shifts to the energies but not discrete quantum energy shifts). This is because the form of the general interaction potential $V(\rho,\varphi,z)$ we have adopted in (5).

Yet, auxiliary constraints on the ambiguity parameters (see (21) for the harmonic oscillator and (36) for the Coulombic) are observed mandatory to

secure the reality of E. Hereby, if $m = 0$ is considered in (21) and (36) as a reference test, then one would observe that only the Gora's and Williams' ambiguity parametric set ($\beta = \gamma = 0$, $\alpha = -1$) fails to provide real energies (i.e., $\sqrt{3/2 - (\zeta - \beta)} \in \mathbb{C}$) . We contemplate that more auxiliary constraints on the ambiguity parameters should be anticipated for different, though exactly solvable, power-law type radial masses (within our methodical proposal, of course). Furthermore, the spectral signatures of different $\tilde{V}(z)$ interactions on the overall spectrum are also reported. Namely, the spectral signatures of (i) two impenetrable walls at $z = 0$ and $z = L$ provided by the potential well $\tilde{V}(z) = 0$ for $0 < z < L$ and $\tilde{V}(z) = \infty$ elsewhere, (ii) a Morse type [31] interaction $\tilde{V}(z) = D\ \left(e^{-2\epsilon z} - 2e^{-\epsilon z}\right);\ D > 0$, and (iii) a trigonometric Rosen-Morse [32] potential $\tilde{V}(z) = U_{\circ} \cot^2(\pi z/d)\,; z \in [0, d]$.

References

[1] O Von Roos, Phys. Rev. **B 27** (1983) 7547

[2] A Puente, M Casas, Comput. Mater Sci. **2** (1994) 441

[3] A R Plastino, M Casas, A Plastino, Phys. Lett. **A281** (2001) 297.

[4] A Schmidt, Phys. Lett. **A 353** (2006) 459.

A Schmidt, J Phys **A:** Math. Theor.**42** **(**2009**)** 245304.

[5] S H Dong, M Lozada-Cassou, Phys. Lett. **A 337** (2005) 313.

[6] I O Vakarchuk, J. Phys. **A**; Math. Gen. **38** (2005) 4727.

[7] C Y Cai, Z Z Ren, G X Ju, Commun. Theor. Phys. **43** (2005) 1019.

[8] B Roy, P Roy, Phys. Lett. **A 340** (2005) 70.

[9] B Gonul, M Kocak, Chin. Phys. Lett. **20** (2005) 2742.

[10] A de Souza Dutra, C A S Almeida, Phys Lett. **A 275** (2000) 25.

[11] O Mustafa, S.Habib Mazharimousavi, Int. J. Theor. Phys **46** (2007) 1786.

[12] S. Cruz y Cruz, J Negro, L. M. Nieto, Phys. Lett. **A 369** (2007) 400.

[13] S. Cruz y Cruz, O Rosas-Ortiz, J Phys **A**: Math. Theor. **42** (2009) 185205

[14] J Lekner, Am. J. Phys. **75** (2007) 1151

[15] C Quesne, V M Tkachuk, J. Phys. **A**: Math. Gen. **37** (2004) 4267.

[16] L Jiang, L Z Yi, C S Jia, Phys. Lett. **A 345** (2005) 279.

[17] O Mustafa, S H Mazharimousavi, Phys. Lett. **A 358** (2006) 259.

[18] J I Diaz, J Negro, L M Nieto, O Rosas-Ortiz, J Phys **A**; Math. Gen. **32** (1999) 8447

[19] A D Alhaidari, Phys. Rev. **A 66** (2002) 042116.

[20] O Mustafa, S H Mazharimousavi, J. Phys. **A**: Math. Gen. **39** (2006) 10537.

S H Mazharimousavi, O Mustafa; SIGMA **6,** (2010) 088

[21] B Bagchi, A Banerjee, C Quesne, V M Tkachuk, J. Phys. **A**; Math. Gen. **38** (2005) 2929.

[22] J Yu, S H Dong, Phys. Lett. **A 325** (2004) 194.

[23] C Quesne, Ann. Phys. **321** (2006) 1221.

[24] T Tanaka, J. Phys. **A**; Math. Gen. **39** (2006) 219.

[25] A de Souza Dutra, J. Phys. **A**; Math. Gen. **39** (2006) 203.

[26] O Mustafa, S H Mazharimousavi, Czech. J. Phys **56** (2006) 297

[27] O Mustafa, S H Mazharimousavi, Phys. Lett. **A 357** (2006) 295

[28] O Mustafa, S H Mazharimousavi, J Phys **A:** Math. Theor.**41 (**2008**)** 244020

R Koc, G Sahinoglu, M Koca, Eur. Phys. J. **B48** (2005) 583

[29] O Mustafa, S H Mazharimousavi, Phys. Lett. **A 373** (2009) 325

O Mustafa, S H Mazharimousavi Phys. Scr**. 82** (2010) 065013

[30] J. M. Lėvy-Leblond, Phys. Rev. **A52** (1995) 1845.

[31] O. Mustafa, J Phys **A:** Math. Theor.**43 (**2010**)** 385310

[32] Z Q Ma, A. Gonzalez-Cisneros, B W Xu, S H Dong, Phys. Lett. **A 371** (2007) 180